An Observation on the Sommerfeld-Integral Representation of the Electric Dyadic Green's Function for Layered Media

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Abstract —**The electric dyadic Green's function for layered dielectrics is discussed. It is well known that for the free-space electric dyadic Green's function $\overline{\overline{G}}_0$, evaluation of the electric field at observation points within the source region requires specification of a "principal volume" along with the corresponding depolarizing dyad $\overline{\overline{L}}$. Special considerations are invoked for layered background media which are appropriate for the electromagnetics of integrated electronics. It is shown that use of the Sommerfeld-integral representation of the electric dyadic Green's function leads to an innate choice for the depolarizing dyad. A corresponding principal volume is subsequently identified; it is demonstrated that there exists an alternative choice for this excluding region which leads to the same depolarizing dyad.**

I. INTRODUCTION

There is an increasing interest in the study of optical and electronic circuits immersed in a layered dielectric surround. Conventional differential-operator formulations for the fields within these circuit devices are rendered ineffective due to the inseparability of the applicable boundary conditions for structures having practical shapes. An integral-operator formulation, based on the identification of equivalent volume polarization currents, circumvents this difficulty. Construction of the integral operator requires knowledge of the Green's function for the layered surround.

A general development of the Hertzian potential Green's dyad $\overline{\overline{G}}$ for layered dielectrics has been discussed by Bagby and Nyquist [1]. Based on the classical method of Sommerfeld [2], the Hertzian potential dyadic Green's function was shown to have scalar components represented by 2-D spectral integrals. As asserted by Yaghjian [3], the singularity of $\overline{\overline{G}}$ is seen to arise from that part of the dyad which is the Green's function $\overline{\overline{G}}^p$ ("principal" Green's dyad) for the unbounded-space problem. In Section II, the development in [1] is altered slightly so that identification of a natural depolarizing dyad $\overline{\overline{L}}$, relevant to the Green's dyad $\overline{\overline{G}}^e$ for the electric field, may subsequently be made in Section III.

Finally, the electric field is expressed in the standard form as a volume integration of the scalar product of $\overline{\overline{G}}^e$ with the electric current source $\boldsymbol{J}$. The volume of integration extends over the support of the current density but excludes the singularity point of $\overline{\overline{G}}^e$. The excluding region is identified as the "principal volume" which corresponds to the preferred choice of the depolarizing dyad as tabulated in [3]. An alternative excluding region, which is shown to be equivalent, is suggested to be useful in practice due to its simple form.

II. HERTZIAN POTENTIAL GREEN'S DYAD

In this section, the Hertzian potential dyadic Green's function is developed for the trilayered structure depicted in Fig. 1. A film layer of thickness t and refractive index n_f is deposited over a substrate region $(y < -t)$ which is characterized by index of refraction n_s. The region $(y > 0)$ is the *cover* with refractive index n_c in which electric current density $\boldsymbol{J}$, maintaining electromagnetic fields in all three regions, is immersed. All media are understood to possess limitingly small dissipation. Although the ensuing analysis may be generalized for a structure having any number of dielectric layers with embedded currents, the situation in Fig. 1 serves for the purpose of illustration, and provides a useful model for the background of practical electronic and optical integrated circuits.

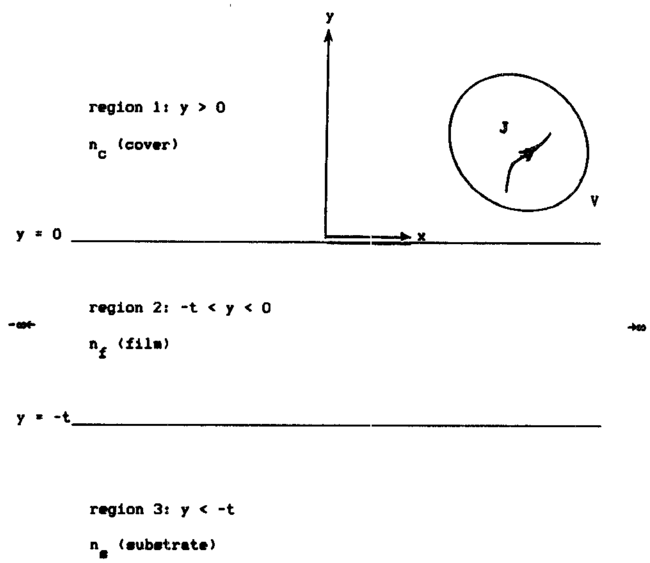

Fig. 1. Trilayered dielectric structure with sources in the cover.

Subsequent analysis assumes: i) time harmonic $(e^{j\omega t})$ dependence of the solutions to Maxwell's equations and ii) all integrals with unspecified limits span the entire space. The Hertzian potential subject to the Lorentz gauge satisfies the Helmholtz equation

$$\left(\nabla^2 + k_i^2\right)\Pi_i = -\boldsymbol{J}/j\omega\epsilon_c \tag{1}$$

in each region $(i = s, f, c$ for substrate, film, cover). Formal operation on (1) with the 2-D Fourier transform

$$F\{\cdot\} = \iint \{\cdot\}\, e^{-j\lambda\cdot r}\, dx\, dz \tag{2}$$

where $\lambda = \hat{x}\xi + \hat{z}\zeta$, reduces (1) to the ordinary differential equation

$$\left(\partial^2/\partial y^2 - p_i^2\right)\pi(\lambda; y) = -j(\lambda; y)/j\omega\epsilon_c \tag{3}$$

where $\pi = F\{\Pi\}$, $j = F\{\boldsymbol{J}\}$, and $p_i^2 = \xi^2 + \zeta^2 - k_i^2$. Solution of (3) is elementary, and may be written as a sum of primary scattered parts. Thus this decomposition is

$$\pi(\lambda; y) = \delta_{ic}\left\{\int_V g^p(\lambda; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\, dV'\right\} + W_i^+(\lambda)\, e^{p_i y} + W_i^-(\lambda)\, e^{-p_i y} \tag{4}$$

where $g^p(\lambda; y, \boldsymbol{r}') = e^{-j\lambda\cdot r'} e^{-p_c|y-y'|}/2p_c$, and δ_{ic} is a Kronecker delta. The coefficients $W_i^{\pm}$ are determined by satisfying the appropriate boundary conditions [1] across the dielectric interfaces and as $y \to \pm\infty$.

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Inversion of the transform-domain potentials yields the solution to (1) with the potential in the *cover* region given by

$$\Pi_c(\boldsymbol{r}) = \frac{1}{(2\pi)^2}\iint e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}\left\{\int_V g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right\} d\xi\, d\zeta + \int_V \overline{\overline{G}}^r(\boldsymbol{r}|\boldsymbol{r}')\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'. \quad (5)$$

The reflected dyad $\overline{\overline{G}}^r(\boldsymbol{r}|\boldsymbol{r}')$ has scalar components $G_{\alpha\beta}(\boldsymbol{r}|\boldsymbol{r}')$ represented by 2-D Sommerfeld integrals of the generic form

$$G_{\alpha\beta}(\boldsymbol{r}|\boldsymbol{r}') = \frac{1}{(2\pi)^2}\iint C_{\alpha\beta}(\boldsymbol{\lambda})\, e^{j\boldsymbol{\lambda}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\frac{e^{-p_c(y+y')}}{2p_c}\,d\xi\, d\zeta. \quad (6)$$

Each of the coefficients $C_{\alpha\beta}(\boldsymbol{\lambda})$ is a well-behaved function of $\boldsymbol{\lambda}$ in the entire $\xi-\zeta$ plane; hence derivatives of $\overline{\overline{G}}^r$ may be obtained by formally differentiating under the spectral integral. Special attention is required in determining derivatives of the principal part of Π.

The spectral integral on the right side of (5) represents the primary part Π^p of the Hertzian potential. It is shown in the Appendix that under the assumption that $\boldsymbol{J}$ and $\nabla\cdot\boldsymbol{J}$ are continuous and of compact support in V, derivatives up to second order of Π^p may be obtained by formally differentiating under the spectral integral. Therefore,

$$\nabla\nabla\cdot\Pi^p(\boldsymbol{r}) = \frac{1}{(2\pi)^2}\iint\nabla\nabla\cdot\left\{e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}\left[\int_V g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right]\right\} d^2\boldsymbol{\lambda} \quad (7a)$$

$$= \frac{1}{(2\pi)^2}\left(\iint\nabla\nabla\cdot\left\{e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}\left[\int_{y'<y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right]\right\} d^2\boldsymbol{\lambda} + \iint\nabla\nabla\cdot\left\{e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}\left[\int_{y'>y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right]\right\} d^2\boldsymbol{\lambda}\right) \quad (7b)$$

where the spatial integration has been split into regions in which g^p is continuously differentiable. Tangential derivatives (i.e., derivatives with respect to x and z) of the bracketed term in (7b) operate only on $e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}$. However, performing the derivatives with respect to y demands additional considerations. Appropriate use of Leibnitz's rule [4, pp. 321–325] for differentiation under the integral sign reveals that

$$\frac{\partial}{\partial y}\left\{\int_{y'<y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right\}$$

$$= \int_{y'<y}\frac{\partial}{\partial y}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y}\frac{\partial}{\partial y}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \iint e^{-j\boldsymbol{\lambda}\cdot\boldsymbol{r}'}\left\{\left[\frac{e^{-p_c(y-y')}}{2p_c} - \frac{e^{-p_c(y'-y)}}{2p_c}\right]\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\right\}\Bigg|_{y'=y} dx'\,dz'$$

$$= \int_{y'<y}\frac{\partial}{\partial y}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y}\frac{\partial}{\partial y}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'.$$

A subsequent differentiation with respect to y yields

$$\frac{\partial^2}{\partial y^2}\left\{\int_{y'<y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y} g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right\}$$

$$= \int_{y'<y}\frac{\partial^2}{\partial y^2}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y}\frac{\partial^2}{\partial y^2}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \iint e^{-j\boldsymbol{\lambda}\cdot\boldsymbol{r}'}\left\{\left[\frac{\partial}{\partial y}\frac{e^{-p_c(y-y')}}{2p_c} - \frac{\partial}{\partial y}\frac{e^{-p_c(y'-y)}}{2p_c}\right]\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\right\}\Bigg|_{y'=y} dx'\,dz'$$

$$= \int_{y'<y}\frac{\partial^2}{\partial y^2}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' + \int_{y'>y}\frac{\partial^2}{\partial y^2}g^p(\boldsymbol{\lambda}; y, \boldsymbol{r}')\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV' - \iint\frac{\boldsymbol{J}(x', y, z')}{j\omega\epsilon_c}e^{-j\boldsymbol{\lambda}\cdot\boldsymbol{r}'}\,dx'\,dz'$$

so that (7b) becomes

$$\nabla\nabla\cdot\Pi^p(\boldsymbol{r}) = \iint\left[\int_V\overline{\overline{g}}(\boldsymbol{\lambda}; \boldsymbol{r}, \boldsymbol{r}')\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right]d^2\boldsymbol{\lambda} - \frac{1}{(2\pi)^2}\iint\left\{\iint\frac{\hat{y}J_y(x', y, z')}{j\omega\epsilon_c}e^{-j\boldsymbol{\lambda}\cdot\boldsymbol{r}'}\,dx'\,dz'\right\}e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}}\,d^2\boldsymbol{\lambda}$$

$$= \iint\left[\int_V\overline{\overline{g}}(\boldsymbol{\lambda}; \boldsymbol{r}, \boldsymbol{r}')\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')}{j\omega\epsilon_c}\,dV'\right]d^2\boldsymbol{\lambda} - \overline{\overline{L}}\cdot\boldsymbol{J}(\boldsymbol{r})/j\omega\epsilon_c \quad (8)$$

where $\overline{\overline{L}} = \hat{y}\hat{y}$ and the dyad $\overline{\overline{g}}$ is given by the expression

$$\overline{\overline{g}}(\boldsymbol{\lambda}; \boldsymbol{r}, \boldsymbol{r}') = \begin{cases}\nabla\nabla\left[e^{j\boldsymbol{\lambda}\cdot(\boldsymbol{r}-\boldsymbol{r}')}e^{-p_c(y-y')}/8\pi^2 p_c\right], & y'<y\\ \nabla\nabla\left[e^{j\boldsymbol{\lambda}\cdot(\boldsymbol{r}-\boldsymbol{r}')}e^{-p_c(y'-y)}/8\pi^2 p_c\right], & y'>y.\end{cases} \quad (9)$$

The term $\overline{\overline{L}}\cdot\boldsymbol{J}$ was extracted from exploitation of the Fourier inversion theorem [5, p. 315], and is found to correspond exactly with that exposed in [3] for a "pillbox" principal volume. The form of $\overline{\overline{g}}$ suggests that the "slice" exclusion in Fig. 2 might be a more natural principal volume pertaining to $\overline{\overline{L}}$. This assertion is verified in part B of Section III.

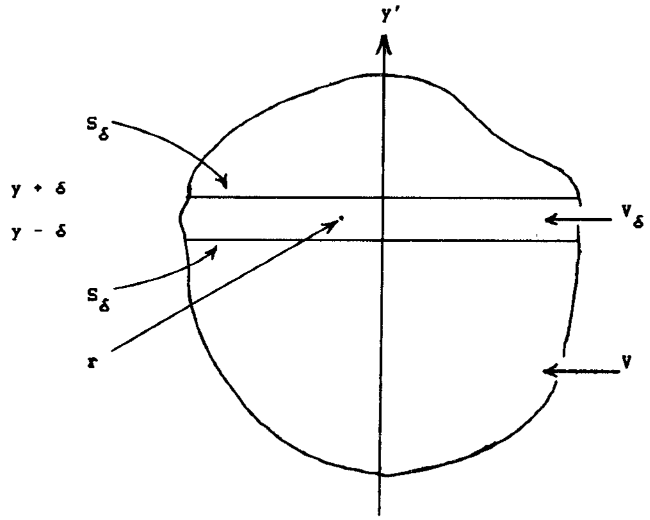

Fig. 2. A "slice" principal volume excluding the singularity point of the electric dyadic Green's function; closed surface S_δ is the boundary of the slice volume.

III. Electric Dyadic Green's Function

A. Development of the Principal Dyad

The electric field E is related to the Hertzian potential by $E=(k_c^2+\nabla\nabla\cdot)\Pi$. Using (8), the principal part of the field may be written as

$$E^p(r) = -j\omega\mu_0 \iint \left\{ \int_V \bar{\bar{g}}^e(\lambda; r, r')\cdot J(r')\,dV' \right\} d^2\lambda - \bar{\bar{L}}\cdot J(r)/j\omega\epsilon_c \quad (10)$$

where the dyad $\bar{\bar{g}}^e = \bar{\bar{g}}/k_c^2 + (1/4\pi^2)\bar{\bar{I}}g^p e^{j\lambda\cdot r}$.

Equation (10) is a useful expression for the principal part of the electric field due to the simple nature of the integrand appearing in the volume integral. However, (10) is not written in the standard form as a volume integration of the scalar product of a Green's dyad with the electric current density. The depolarizing dyad $\bar{\bar{L}}$ has manifested itself naturally. Recognizing that the corresponding "principal volume" is a pillbox [3] yields the standard form for the electric field:

$$E^p(r) = -j\omega\mu_0 \lim_{v\to 0}\int_{V-v} \bar{\bar{G}}^e(r,r')\cdot J(r')\,dV' - \bar{\bar{L}}\cdot J(r)/j\omega\epsilon_c \quad (11)$$

where v is a pillbox excluding the singularity of $\bar{\bar{G}}^e$ at r and $\bar{\bar{G}}^e$ is given by

$$\bar{\bar{G}}^e(r,r') = \begin{cases} \left(\bar{\bar{I}}+\nabla\nabla/k_c^2\right)\iint e^{j\lambda\cdot(r-r')}\dfrac{e^{-p_c(y-y')}}{2(2\pi)^2 p_c}\,d^2\lambda, & y'<y \\ \left(\bar{\bar{I}}+\nabla\nabla/k_c^2\right)\iint e^{j\lambda\cdot(r-r')}\dfrac{e^{-p_c(y'-y)}}{2(2\pi)^2 p_c}\,d^2\lambda, & y'>y. \end{cases} \quad (12)$$

B. Equivalence of Principal Volumes

The principal volume v in (11) was identified to be a pillbox as tabulated in [3]. A more useful, and equivalent, exclusion is the "slice" volume shown in Fig. 2. Starting with the common representation for the free-space Green's function $\psi(r|r') = e^{-jk_c|r-r'|}/4\pi|r-r'|$, it may be shown that for a "slice" principal volume, the correction term $E^c(r)$ for the electric dyadic Green's function for field points in the source region is

$$E^c(r) = -\frac{1}{j\omega\epsilon_c}\lim_{\delta\to 0}\int_{S_\delta} \nabla'\psi(r|r')\,\hat{n}'\cdot J(r')\,dS' \quad (13)$$

where S_δ is shown in Fig. 2. The correction term above is now shown to be equivalent to the correction term corresponding to a pillbox principal volume. The surface integral term in (13) is split into integrations over S_1 and S_2 (planes at $y\pm\delta$, respectively). This yields for (13)

$$E^c(r) = -\frac{1}{j\omega\epsilon_c}\lim_{\delta\to 0}\left\{ -\int_{S_1}\nabla'\psi(r|r')|_{y'=y+\delta} \cdot J_y(x', y+\delta, z')\,dx'dz' + \int_{S_2}\nabla'\psi(r|r')|_{y'=y-\delta} J_y(x', y-\delta, z')\,dx'dz' \right\}. \quad (14)$$

As $\delta\to 0$, S_1 approaches S_2 and $J_y(x', y+\delta, z')$ approaches $J_y(x', y-\delta, z')$ due to the smoothness of the boundary of V and the continuity of J at $y'=y$. Thus, (14) simplifies to

$$E^c(r) = \frac{1}{j\omega\epsilon_c}\lim_{\delta\to 0}\int_S \nabla'\psi(r|r')|_{y'=y-\delta}^{y'=y+\delta} J_y(x', y, z')\,dx'dz' \quad (15)$$

where S extends over the $x'-z'$ plane. Expressing $\nabla'\psi$ in Cartesian form as

$$\nabla'\psi(r|r') = (-jk_c - 1/R)\frac{e^{-jk_cR}}{4\pi R^2} \cdot [\hat{x}(x'-x)+\hat{y}(y'-y)+\hat{z}(z'-z)]$$

where $R=|r-r'|$, it is found that

$$\nabla'\psi(r|r')|_{y'=y-\delta}^{y'=y+\delta} = (-jk_c - 1/R)\frac{e^{-jk_cR_\delta}}{4\pi R_\delta^2}\hat{y}2\delta \quad (16)$$

where $R_\delta = [(x-x')^2+(z-z')^2+\delta^2]^{1/2}$. Substitution of (16) into (15) yields

$$E^c(r) = \frac{1}{j\omega\epsilon_c}\lim_{\delta\to 0}\left\{ \hat{y}\int_S 2\delta(-jk_c - 1/R_\delta) \cdot \frac{e^{-jk_cR_\delta}}{4\pi R_\delta^2} J_y(x', y, z')\,dx'dz' \right\}. \quad (17)$$

The integral in (17) may be decomposed into the sum of integrals over $S-C_\nu$ and C_ν. C_ν is a circle centered at (x,z) with radius ν. As $\delta\to 0$, integration over $S-C_\nu$ vanishes. If ν is chosen sufficiently small, then $J_y(x', y, z')\approx J_y(r)$ and $e^{-jk_cR_\delta}\approx 1$ so that (17) becomes

$$E^c(r) = -\frac{1}{j\omega\epsilon_c}\hat{y}J_y(r)\lim_{\delta\to 0}\left\{ \delta\int_{C_\nu}\frac{jk_c+1/R_\delta}{2\pi R_\delta^2}\,dx'dz' \right\} \quad (18a)$$

$$= -\frac{1}{j\omega\epsilon_c}\hat{y}J_y(r)\lim_{\delta\to 0}\left\{ \delta\int_0^{2\pi}d\varphi\int_0^{\nu}\frac{jk_c+1/r_\delta}{2\pi r_\delta^2}\rho\,d\rho \right\} \quad (18b)$$

$$= -\frac{1}{j\omega\epsilon_c}\hat{y}J_y(r)\lim_{\delta\to 0}\left\{ \delta\int_0^{\nu}\frac{jk_c+1/r_\delta}{r_\delta^2}\rho\,d\rho \right\} \quad (18c)$$

where $r_\delta = (\rho^2 + \delta^2)^{1/2}$. In going from (18a) to (18b), integration over C_ν has been transformed to polar coordinates. Performing the angular integration is trivial and yields (18c). Noting that the integrand in (18c) is a perfect differential, the term in braces becomes

$$\delta \int_0^\nu \frac{jk_c + 1/r_\delta}{r_\delta^2} \rho \, d\rho = \delta \left\{ jk_c \ln(\rho^2+\delta^2)^{1/2} - (\rho^2+\delta^2)^{-1/2} \right\} \Big|_0^\nu$$
$$= \delta \left\{ jk_c \left[\ln(\nu^2+\delta^2)^{1/2} - \ln\delta \right] + \left[1/\delta - (\nu^2+\delta^2)^{-1/2} \right] \right\}$$
$$= 1 \quad (\text{as } \delta \to 0). \tag{19}$$

Finally, substitution of (19) into (18c) yields

$$\boldsymbol{E}^c(\boldsymbol{r}) = -\frac{1}{j\omega\epsilon_c} \hat{y} J_y(\boldsymbol{r})$$

which is precisely the same correction term appearing in (11). Therefore integration excluding a "slice" principal volume is equivalent to a pillbox exclusion, and evaluation of (11) may use either of these volumes.

IV. Conclusions

In the study of layered-media electromagnetics, Sommerfeld integrals are used to represent scalar components of the electric dyadic Green's function. The principal part of the electric field may be written as a sum of a spectral integral along with a correction term that appears naturally. The spectral integration may be replaced with the more standard volume integral as in (11). Recognition of the depolarizing dyad, which has manifested itself innately, identifies the appropriate principal volume (a pillbox). An equivalent excluding region to the pillbox is suggested to be useful in practice due to its simple form.

Appendix

It is now shown that the differentiation under the spectral integral as in (7) is a legitimate operation. Without loss of generality, justifying this interchange of operations for the following is sufficient:

$$\iint_{\lambda > k_r} \nabla\nabla \cdot \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \int_V g^p(\lambda; y, \boldsymbol{r}') \boldsymbol{J}(\boldsymbol{r}') \, dV' \right] d^2\lambda. \tag{A1}$$

In (A1), k_r is the real part of k_c. Evaluation of p_c is made on the Riemann sheet with $\mathrm{Re}\{p_c\} > 0$.

Assuming that $\boldsymbol{J}$ and $\nabla\cdot\boldsymbol{J}$ are continuous and have compact support in V, use of the vector identity $\nabla\cdot\{\varphi\boldsymbol{A}\} = \varphi\nabla\cdot\boldsymbol{A} + \nabla\varphi\cdot\boldsymbol{A}$ along with the divergence theorem on (A1) yields

$$\iint_{\lambda > k_r} \nabla\nabla \cdot \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \int_V g^p(\lambda; y, \boldsymbol{r}') \boldsymbol{J}(\boldsymbol{r}') \, dV' \right] d^2\lambda$$
$$= \iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \int_V \nabla'\cdot\boldsymbol{J}(\boldsymbol{r}') g^p(\lambda; y, \boldsymbol{r}') \, dV' \right] d^2\lambda$$
$$= \iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \int \boldsymbol{F}\{\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')\} \frac{e^{-p_c|y-y'|}}{2p_c} \, dy' \right] d^2\lambda \tag{A2}$$

where $\boldsymbol{F}\{\nabla\cdot\boldsymbol{J}\}$ is the Fourier transform of $\nabla\cdot\boldsymbol{J}$ as defined by (2).

Next, use $p_c = p_r + jp_i$, where p_r and p_i are the real and imaginary parts of p_c, respectively. The exponential $e^{-p_r|y-y'|}$ is of constant sign for all y. By the generalized first mean value theorem for integrals [4, p. 117], the right side of (21) may be written as

$$\iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \mathrm{Re}\left\{ \boldsymbol{F}\{\nabla'\cdot\boldsymbol{J}(x',\eta,z')\} e^{-jp_i|y-\eta|} \right\} \cdot \int_{y_{\min}}^{y_{\max}} \frac{e^{-p_r|y-y'|}}{2p_c} \, dy' \right] d^2\lambda$$
$$+ j \iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \mathrm{Im}\left\{ \boldsymbol{F}\{\nabla'\cdot\boldsymbol{J}(x',\theta,z')\} e^{-jp_i|y-\theta|} \right\} \cdot \int_{y_{\min}}^{y_{\max}} \frac{e^{-p_r|y-y'|}}{2p_c} \, dy' \right] d^2\lambda \tag{A3}$$

where $y_{\min} < (\eta, \theta) < y_{\max}$ ($\boldsymbol{J} = 0$ for all $y' < y_{\min}$, $y > y_{\max}$). The spatial integration in (22) is trivial and leads to

$$\iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \mathrm{Re}\left\{ \boldsymbol{F}\{\nabla\cdot\boldsymbol{J}(x',\eta,z')\} \cdot e^{-jp_i|y-\eta|} \right\} \frac{\varphi(p_r; y)}{2p_r p_c} \right] d^2\lambda$$
$$+ j \iint_{\lambda > k_r} \nabla \left[e^{j\boldsymbol{\lambda}\cdot\boldsymbol{r}} \mathrm{Im}\left\{ \boldsymbol{F}\{\nabla'\cdot\boldsymbol{J}(x',\theta,z')\} \cdot e^{-jp_i|y-\theta|} \right\} \frac{\varphi(p_r; y)}{2p_r p_c} \right] d^2\lambda \tag{A4}$$

where $\varphi(p_r; y) = \{2 - e^{-p_r(y - y_{\min})} - e^{-p_r(y_{\max} - y)}\}$. Since $\nabla\cdot\boldsymbol{J}$ is continuous and of compact support in V, $\nabla\cdot\boldsymbol{J} \in L^2$ (i.e., the space of square integrable functions). In particular, for each y, $\nabla\cdot\boldsymbol{J}$ is an L^2 function in the $\xi - z$ plane. Using a standard theorem from Fourier transform theory [5, pp. 310–313], the 2-D Fourier transform of $\nabla\cdot\boldsymbol{J}$ is an L^2 function in the $\xi-\zeta$ plane. Thus, $\boldsymbol{F}\{\nabla\cdot\boldsymbol{J}\} = 0(\lambda^{-1-\epsilon})$ as ($\lambda \to \infty$, $\epsilon > 0$). The integrand in (A4) is dominated in magnitude by a function which is independent of r and $0(\lambda^{-2-\epsilon})$. The Weierstrauss M test [4, p. 470] guarantees that the integral in (A4) converges uniformly, whereby a standard theorem from advanced calculus [4, p. 474] justifies the interchange of differentiation and spectral integration.

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